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On the zeros of 6-j coefficients

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Abstract. An extensive computer search for non-trivial zeros of the 6-j coefficients of $SU(2)$ has been performed to find their frequency as a function of the sum of their quantum numbers and of their polynomial degree. For the zeros of degree 2, four of the seven cases already known (taking into account Regge symmetries) of two relations between quantum numbers relevant to a Pell equation have been generalized to deal with only one relation; a new case of two relations and many new cases of three relations, all with polynomial solutions, have been found; but all these formulae only account for one zero out of seven. For degree 3, we found four new cases of two relations and some other cases of three relations with polynomial solutions; with the already known case of a Pell equation, they account for almost half of the zeros. The fast decrease of the number of zeros with the degree and the relative smallness of their quantum number is an indication that there should be no zeros of degree larger than 9.

1. Introduction

In a previous work (Raynal *et al* 1993) we studied the zeros of the 3-j coefficients of $SU(2)$ with a particular emphasis on their repartition. In the past, such studies have usually been associated with a study of the 6-j coefficients of $SU(2)$, due to the similarity of these two problems. But our findings for 3-j coefficients have no equivalent for 6-j coefficients; however, the study of their repartition has a similar interest. Let us summarize previous work on the zeros of 6-j coefficients.

For the 6-j coefficients of $SU(2)$ there exists a class of zeros which have been called 'non-trivial' or 'structural' zeros as opposed to the 'trivial' zeros resulting from a violation of one or more triangle conditions. These non-trivial zeros have been the subject of many studies some years ago, sometimes in relation with other groups (Koozekanani and Biedenharn 1974, Biedenharn and Louck 1981, Van der Jeugt *et al* 1983, De Meyer *et al* 1984, Vanden Berghe *et al* 1984, Van der Jeugt 1992, Vanden Berghe 1994). Bowick (1976) shortened the tables of zeros published by Koozekanani and Biedenharn (1974) for 6-j coefficients, taking into account the symmetries discovered by Regge (1959). These symmetries will be fully taken into account in the present work.

The non-trivial zeros of 6-j coefficients, like those of the 3-j coefficients, have been classified by the minimum length of the single sum expression for the coefficients. This is also the minimum length of the expression of the coefficients in terms of a generalized hypergeometric series (Lindner 1985, Srinivasa Rao 1985, Srinivasa Rao and Rajeswari 1985). The number of terms in this sum minus one will be called the *degree* of the coefficient (in some papers it is called the weight). Thus zeros of degree n ($n > 0$) are

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by definition non-trivial zeros. The zeros of degree 1 of the 6-j coefficients have been studied by Brudno (1985), Brudno and Louck (1985a, b), Bremmer and Brudno (1986), Srinivasa Rao and Rajeswari (1987), and Srinivasa Rao *et al* (1988), but no general expression without some constraint between its parameters has been found. Using the Pell equation, Beyer *et al* (1986) showed that there are infinite series of zeros of degree 2 for 6-j coefficients. Algorithms were obtained by Srinivasa Rao and Chiu (1989). For the zeros of 6-j coefficients of degree 3, Brudno (1987) found a special case for which a Pell equation can be used.

The original motivation for the present paper was an investigation of the distribution of the zeros with respect to their degree n and the sum of the angular momentum quantum numbers involved (this sum is invariant for Regge symmetries). The method of computation is similar to the one used by Raynal *et al* (1993) for the zeros of 3-j coefficients. A first search for the zeros of 6-j coefficients up to a sum of angular momenta equal to 240 gave only two zeros of degree 6, and one of degree 7, 8 and 9. So, it appears that zeros are quite scarce for high degree and an extended search was performed up to a sum of quantum numbers equal to 600. Finally, the search was extended to a sum equal to 1200, but only for the zeros of degree 2 to 8. All the results are used to find which relations between quantum numbers allow the parametrization of families of zeros.

In section 2, notations for 6-j coefficients are summarized with emphasis on Regge symmetry. Some indications of the computation are given. Section 3 is devoted to the zeros of degree 2. The generalization of four of the cases relevant to a Pell equation found by Beyer *et al* (1986), a new case with two relations between quantum numbers and many cases with three relations, only involve about one out of seven zeros. In section 4, the zeros of degree 3 are found to be easier to handle than the zeros of degree 2. Beside the Pell equation already found by Brudno (1987), there are four cases with two linear relations between parameters in which one can give solutions quadratic in some variable and even extract linear solutions. Including cases with three linear conditions, we obtain almost one zero out of two. Results for the zeros of higher degree are discussed in section 5. Some finite families can be found for degrees 4 and 5, but more easily for degree 5. All the zeros found for degrees 5 and more are given in tables.

2. Expression of the 6-j coefficients

The 6-j coefficients of $SU(2)$ are conveniently expressed as a generalized hypergeometric series:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = C\,{}_4F_3[a-e-f, a-b-c, -c+d-e, -b+d-f; \\ -b-c-e-f-1, a-b+d-e+1, a-c+d-f+1; 1] \tag{1}$$

where C is some non-zero factor and ${}_4F_3$ is a Saalschützian generalized hypergeometric series, i.e. the sum of the denominator parameters is equal to the sum of the numerator parameters plus one. This formula holds when the two last denominator parameters are positive and the sum is limited by the smallest number in the symbol defined by Shelepin (1965), which is

$$\begin{vmatrix} -c+d+e & b+d-f & a+e-f & a+b-c \\ -b+d+f & c+d-e & a-b+c & a-e+f \\ -a+e+f & -a+b+c & c-d+e & b-d+f \end{vmatrix}. \tag{2}$$

As for the 3-j coefficient, this expression in terms of a generalized hypergeometric series is not unique (see for example Varshalovich *et al* 1975) and ten other different formulae could be used (Raynal 1979). Whipple (1936) studied these series in conjunction with *well poised* ${}_7F_6$ generalized hypergeometric series, introducing six parameters which are

$$\begin{aligned} x_0 &= -\tfrac{1}{2}(a+d+1) & x_1 &= -\tfrac{1}{2}(b+e+1) & x_2 &= -\tfrac{1}{2}(c+f+1) \\ x_3 &= -\tfrac{1}{2}(a-d) & x_4 &= -\tfrac{1}{2}(b-e) & x_5 &= -\tfrac{1}{2}(c-f) \end{aligned} \tag{3}$$

and obtained symmetries corresponding to permutations of these six parameters with a change of sign for an even number of them. For $SU(2)$, the usual symmetries correspond to a permutation of the pairs (x_0, x_3), (x_1, x_4) and (x_2, x_5) and an even number of changes of sign among (x_3, x_4, x_5), and the Regge symmetries to a permutation of (x_0, x_1, x_2) independent of the permutation of (x_3, x_4, x_5). So, to take into account these symmetries *without introducing unusual fractional numbers and many negative signs, it is convenient* to define a 6-j coefficient by the parameters

$$\begin{aligned} X_a &= a+d & X_b &= b+e & X_c &= c+f \\ Y_a &= a-d & Y_b &= b-e & Y_c &= c-f \end{aligned} \tag{4}$$

with $X_a \geqslant X_b \geqslant X_c \geqslant Y_a \geqslant Y_b \geqslant |Y_c| \geqslant 0$, where only Y_c can be negative. The X and the Y are integer or half-integer. All the 288 equivalent 6-j coefficients are obtained by the product of the permutation of X, the permutation of Y and an even number of changes of sign for Y, but the definition (4) is unique. So, X and Y are a convenient intermediate step to obtaining all the equivalent 6-j coefficients. That any X should be greater than any $|Y|$ is easily seen because $X_c < Y_a$ implies $a - c > d + f$, which excludes any value of b. All values of X and Y are not allowed, due to triangular inequalities, which require $-X_a + X_b + X_c - Y_a - Y_b - Y_c \geqslant 0$.

Using the notation

$$(x)_{-i} = x(x-1)\dots(x-i+1) \qquad (x)_i = x(x+1)\dots(x+i-1) \tag{5}$$

we can write the 6-j coefficient,

$$\begin{aligned} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} &= C\,{}_4F_3[-n, -x, -y, -z; -t, u, v; 1] \\ &= C' \sum_{i=0}^{i=n} (-)^i \frac{n!}{i!(n-i)!} (x)_{-i}(y)_{-i}(z)_{-i}(t)_{i-n}(u)_{n-i}(v)_{n-i} \end{aligned} \tag{6}$$

where

$$0 < n \leqslant x \leqslant y \leqslant z \qquad 0 < u \leqslant v \qquad t = n+x+y+z+u+v-1\,. \tag{7}$$

The relations between these parameters and the X and Y are

$$\begin{aligned} 2X_a &= t+u+v-3 & 2X_b &= t-u+v-1 & 2X_c &= t+u-v-1 \\ 2Y_a &= -n-x+y+z & 2Y_b &= -n+x-y+z & 2Y_c &= -n+x+y-z \end{aligned} \tag{8}$$

and with the quantum numbers

$$\begin{aligned} 2a &= y+z+u+v-2 & 2b &= x+z+v-1 & 2c &= x+y+u-1 \\ 2d &= n+x+u+v-2 & 2e &= n+y+v-1 & 2f &= n+z+u-1\,. \end{aligned} \tag{9}$$

In the presentation of the results, we often consider the parameters (x, y, z) as equivalent (without inequality relations) and the same holds for (u, v). Also, we permute (b, c) with (e, f) in writing the coefficients when the result is more symmetric.

In his work, Bowick (1976) used notations introduced by Jahn and Howell (1959) based on the reversed series

$${}_4F_3[1-n+t, -n, 1-n-u, 1-n-v; 1-n+x, 1-n+y, 1-n+z; 1]$$

where $J_m = 1 - n + t$, which turns out to be $a + b + c + 2$, is chosen to classify the coefficients, with the assumption that it is larger than the three other similar expressions. There are no zeros if J_m is a prime number. In this work, we prefer to classify the 6-j coefficients with respect to their degree n and

$$N = 2(a + b + c + d + e + f) = 2(X_a + X_b + X_c) = 3t + u + v - 5 \tag{10}$$

(twice the sum because the sum itself can be integer or half-integer).

To obtain the zeros for a fixed value of n and of N:

- First, we choose t and the sum $s_1 = u + v$ which must be at least 2 and less than $\frac{1}{4}N - 3n + 3$ because $t = n + x + y + z + s_1 - 1$ and $x, y, z \geqslant n$.
- After, we choose the sum $s_2 = x + y$ and z with $s_2 + z = t - s_1 - n + 1$ and $2z \geqslant s_2$.
- Then, we choose x and y such that $x + y = s_2$ and $x \leqslant y \leqslant z$. At this stage, for $n = 1$, we can check if xyz can be divided by t to avoid more computations. Similar verifications can be performed for larger values of n, but they become inefficient for too large values.
- The last step is the choice of u and v with $s_1 = u + v$ and $u \leqslant v$.

For large values of n, the evaluation of the generalized hypergeometric series for every value of u and v turns out to be very long. Instead we calculated these series only for $u = u_m$, $v = v_m$ (where u_m is the maximum value of u and $v_m = u_m$ or $v_m = u_m + 1$) and for $u = u_m - 1$, $v = v_m + 1$ to use a recurrence relation for fixed values of the other parameters. This recurrence turned out to be very stable, going towards small values of u, but not in the other direction. Detection of zeros is done by comparison of the value given by the recurrence to the one obtained by changing the sign between the two terms involved.

We have performed a complete search of the zeros of all degrees n up to $N = 1200$. Afterwards, we continued the search up to $N = 2400$ for the degrees $n = 2$ to $n = 8$ which seemed of special interest. We will give between parenthesis the number of zeros for $N \leqslant 600$, for $600 < N \leqslant 1200$ and $1200 < N \leqslant 2400$. The number of zeros is given with more details in table 1 as a function of n and N.

For $N \leqslant 1200$, there are 105 477 zeros of degree 1 with a maximum number of 321 for $N = 1126$ and 300 for $N = 1184$.

3. Zeros of degree 2

There are 16 252 zeros of degree 2 for $N \leqslant 2400$ (1254, 3713, 11 285) with a maximum number of 23 for $N = 1555$. For this degree, the condition is

$$xyz(x-1)(y-1)(z-1) + tuv(t-1)(u+1)(v+1) = 2xyz(t-1)(u+1)(v+1) \tag{11}$$

with $t = x + y + z + u + v + 1$. It is of second degree in the parameters, but if one of the parameters (x, y, z) or (t, u, v) is eliminated the condition is of fourth degree in the two other parameters.

There are many pairs of zeros with four identical parameters. If the sets (t, u, v) or the sets (x, y, z) are identical, the relation between the other parameters of these zeros is

$$\begin{aligned}
x' + y' &= x + y \qquad & (y' - x')^2 &= (x + y)^2 + 4(x-1)(y-1) - \frac{8(t-1)(u+1)(v+1)}{z-1} \\
t' - u' &= t - u \qquad & (t' + u')^2 &= (t - u)^2 - 4(t-1)(u+1) + \frac{8xyz}{v} \\
u' + v' &= u + v \qquad & (u' - v')^2 &= (u + v)^2 + 4(u+1)(v+1) - \frac{8xyz}{t}.
\end{aligned} \tag{12}$$

Table 1. The number of zeros of degree n for 6-j coefficients as a function of $N = 2(a+b+c+d+e+f)$.

N	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9
1–120	146	16	2	0	0	0	0	0	0
121–240	985	127	30	11	4	0	0	0	0
241–360	2555	240	66	17	7	1	0	0	0
361–480	4660	385	81	12	2	1	1	1	1
481–600	7295	486	82	13	7	0	1	1	0
601–720	10352	593	104	9	11	2	2	0	0
721–840	13894	651	91	23	4	1	0	0	0
841–960	17640	817	118	21	4	0	0	0	0
961–1080	21600	783	97	9	4	1	0	0	0
1081–1200	26350	869	147	16	6	0	0	1	0
1201–1320		936	119	11	1	0	0	0	
1321–1440		969	101	14	1	0	1	0	
1441–1560		1110	96	8	1	0	0	0	
1561–1680		1019	110	6	2	1	0	0	
1681–1800		1164	123	10	0	0	1	0	
1801–1920		1052	113	10	2	0	0	0	
1921–2040		1152	100	7	0	0	0	0	
2041–2160		1349	119	7	1	0	0	0	
2161–2280		1229	116	8	0	0	0	0	
2281–2400		1305	124	5	1	0	0	0	

For a pair of zeros, one with the parameters (x, y, z) and (t, u, v), the other with the parameters (x', y, z) and (t, u', v), the relation is

$$\begin{aligned} x' &= 1 - x - \frac{2(y+z+v-t+1)(t-1)(v+1)(yz+tv)}{yz(y-1)(z-1)+2yz(t-1)(v+1)+tv(t-1)(v+1)} \\ u' &= -1 - u - \frac{2(y+z+v-t+1)yz[(y-1)(z-1)+(t-1)(v+1)]}{yz(y-1)(z-1)+2yz(t-1)(v+1)+tv(t-1)(v+1)} \end{aligned} \tag{13}$$

with the condition that x' and u' should be integers greater than or equal to 2 and 1, respectively. If the parameters of the second zero are (x', y, z) and (t', u, v), the relation is

$$\begin{aligned} x' &= 1 - x + \frac{2(y+z+u+v+1)(u+1)(v+1)(yz-uv)}{yz(y-1)(z-1)-2yz(u+1)(v+1)+uv(u+1)(v+1)} \\ t' &= 1 - t + \frac{2(y+z+u+v+1)yz[(y-1)(z-1)-(u+1)(v+1)]}{yz(y-1)(z-1)-2yz(u+1)(v+1)+uv(u+1)(v+1)} \end{aligned} \tag{14}$$

with the condition that x' and t' should be integers greater than or equal to 2 and 9, respectively. There are six different possibilities for (13) which conserve the value of t and three possibilities for (14) which do not conserve t and can generate large quantum numbers. Recurrently applying these relations to the 16 252 zeros found for $N \leqslant 2400$, 4035 of them generate no other zero, 2145 generate only one zero and 1005 only two zeros; one finds 1308 sets of 4–60 zeros including 3902 zeros beyond $N = 2400$. The first two zeros generate no others and the third zero belongs to a set of 26 zeros. The relations (13) and (14) were very useful in finding the families (49) and (50).

3.1. One relation between quantum numbers

There are eleven conditions like $xy = tu$ or $xy = uv$ for which two degrees factorize in the condition (11). The coefficient of z^2 is of the second degree for a condition like $xy = tu$

but only linear for a condition like $xy = uv$. Furthermore, this linear coefficient can also be factorized in the following four cases.

Case 1: $xy = uv$. There are 226 such zeros for $N \leqslant 2400$ (34, 47, 145). The condition (11) can be written as

$$uv(x+y+u+v)[(u+1)(v+1)(x+y+u+v+1) - z(z-1)] = 0 . \qquad (15)$$

With $x = gik$, $y = gjl$, $u = gjk$, $v = gil$ (where g is a common factor and where i and j have no common divisors, as do k and l), it is a Pell equation for k and z with fixed values of i, j and l,

$$[(gil+1)\{(i+j)(2gjk+gjl+1)+j\}]^2 - j(i+j)(gil+1)[2z-1]^2$$
$$= (gil+1)(gjl-1)\{(i+j)(gil+1)(gijl+gj^2l-i+j)+ij\} . \qquad (16)$$

Three other Pell equations are obtained by exchanging k with l and i with j, k with i and l with j or by the product of these two operations. Exchange of (i, j) with (k, l) is only a permutation of (x, y) or (u, v). Each zero belongs to twelve finite families, six with fixed values of i and j and six with fixed values of k and l. Due to the simplicity of their derivation and their similarity, we postpone their demonstration to the case of the zeros of degree 3 with $x = u+2$ and $y = v+2$ for which two of them have been given by Brudno (1987). Many of these families reduce to only one zero. For nine of the zeros found, they generate no others but one of these families involves 29 zeros.

There are 54 zeros (14, 13, 27) with $i = j = 1$. They have two linear conditions between quantum numbers: $x = u$ and $y = v$. In this case the 6-j coefficient can be written as

$$\begin{Bmatrix} a+b-2 & a & b \\ c+\frac{1}{2} & c & c \end{Bmatrix} \qquad \begin{matrix} x = a-\frac{1}{2} & y = b-\frac{1}{2} & z = -a-b+2c+2 \\ u = a-\frac{1}{2} & v = b-\frac{1}{2} & t = a+b+2c+1 \end{matrix} \qquad (17)$$

and taking into account that $2b$ is odd, the Pell equation (16) can be rewritten as

$$[8c-4a-4b+6]^2 - (2b+1)[4a+2b]^2 = -(2b)^2(2b-3) . \qquad (18)$$

This is case 5 and Pell equation III for Beyer *et al* (1986). If $a \neq b$, each zero belongs to two different sequences of solutions of a Pell equation. If $a = b$, the zero belongs to a polynomial family (all the parameters are expressed as a polynomial of r) which is

$$x = y = u = v = r^2+3r+1 \qquad z = (2r+5)(r+1)^2 \qquad (19)$$

starting from $r = 1$ (up to $r = 5$ for $N \leqslant 2400$) and to a sequence of solutions of a Pell equation.

Case 2: $(x-1)(y-1) = (u+1)(v+1)$. There are 318 such zeros (58, 93, 167). The condition (11) can be written as

$$(u+1)(v+1)(x+y+u+v+1)$$
$$\times[uv(x+y+u+v+1) - z(z+2x+2y+2u+2v+1)] = 0 . \qquad (20)$$

With $x = gik+1$, $y = gjl+1$, $u = gjk-1$, $v = gil-1$, it is a Pell equation for k and z with fixed values of i, j and l which differs from the Pell equation (16) of case 1 by

$$i \Leftrightarrow j \qquad z \Rightarrow z+g(i+j)(k+l)+1 . \qquad (21)$$

Therefore, for any zero of case 1 with $x_1, \ldots$ there is a zero of case 2 with $x_2, \ldots$ such that

$$\begin{matrix} x_2 = v_1+1 & y_2 = u_1+1 & z_2 = 2z_1 - t_1 \\ u_2 = y_1-1 & v_2 = x_1-1 & t_2 = z_1 . \end{matrix} \qquad (22)$$

For the two first zeros of case 1, $z_2 = 1$ and the corresponding zeros are of degree 1. The finite families of the two cases are equivalent only if $i = j = 1$. For 14 zeros, the finite families generate no others but one of these families involves 156 zeros.

There are 70 zeros (19, 19, 32) with $i = j = 1$. They have two linear conditions between quantum numbers: $x = u + 2$ and $y = v + 2$. In this case the 6-j coefficient can be written as

$$\begin{Bmatrix} a+b-2 & a & b \\ c-\frac{1}{2} & c & c \end{Bmatrix} \qquad \begin{matrix} x = a+\frac{1}{2} & y = b+\frac{1}{2} & z = -a-b+2c+2 \\ u = a-\frac{3}{2} & v = b-\frac{3}{2} & t = a+b+2c+1\,. \end{matrix} \tag{23}$$

This is case 6 and Pell equation III of Beyer *et al* (1986) with $[8c + 4a + 4b + 2]$ instead of $[8c - 4a - 4b + 6]$ in equation (18). There is a polynomial family corresponding to the family (19). The correspondence between case 1 and case 2 is

$$\begin{Bmatrix} a+b-2 & a & b \\ c+\frac{1}{2} & c & c \end{Bmatrix}_{(1)} = 0 \Rightarrow \begin{Bmatrix} a+b-2 & a & b \\ c-a-b & c-a-b+\frac{1}{2} & c-a-b+\frac{1}{2} \end{Bmatrix}_{(2)} = 0 \tag{24}$$

Case 3: $xy = u(v+1)$. There are 494 such zeros (84, 129, 281). The condition (11) can be written as

$$u(v+1)(x+y+u+v+1)[(u+1)v(x+y+u+v) - z(z+2u+1)] = 0\,. \tag{25}$$

With $x = gik, y = gjl, u = gjk, v = gil - 1$, it is a Pell equation for k and z with fixed values of i, j and l or a different Pell equation for l and z with fixed values of i, j and k:

$$\begin{aligned} &[\{g(i+j)l - 1\}(2gijk + gijl + i - j)]^2 - ij(gil + gjl - 1)[2z + 2gjk + 1]^2 \\ &\qquad = (gil-1)(gjl-1)\{(i+j)[(gjl-1)(gi^2l + gijl - 3i - j) - 2i] + ij\} \\ &[(gjk+1)\{(i+j)(2gil + gik - 1) - l\}]^2 - i(i+j)(gjk+1)[2z + 2gjk + 1]^2 \\ &\qquad = (gik-1)(gjk+1)\{(i+j)(gjk+1)(gi^2k + gijk + i - j) + ij\}\,. \end{aligned} \tag{26}$$

Two other Pell equations are obtained by exchanging k with i and l with j. Each zero belongs to twelve finite families, many of them with only one zero. For 38 of the zeros found, they generate no others but one of these families involves 99 zeros.

There are 116 zeros (25, 24, 67) with $i = j = 1$. They have two linear conditions between quantum numbers: $x = u$ and $y = v + 1$. In this case the 6-j coefficient can be written as

$$\begin{Bmatrix} a+b-2 & a & b \\ c & c & c-\frac{1}{2} \end{Bmatrix} \qquad \begin{matrix} x = a-\frac{1}{2} & y = b & z = -a-b+2c+\frac{3}{2} \\ u = a-\frac{1}{2} & v = b-1 & t = a+b+2c+\frac{1}{2}\,. \end{matrix} \tag{27}$$

The Pell equations (26) are, respectively, for a and c, and for b and c, taking into account that $2a$ is odd:

$$\begin{aligned} &[4c - 2a + 3]^2 - (2b-1)[2a + b - 1]^2 = -(b-1)^2(2b-5) \\ &[8c - 4a + 6]^2 - (2a+1)[2a + 4b - 4]^2 = -(2a)^2(2a-3) \end{aligned} \tag{28}$$

(respectively, case 3, Pell equation I and case 4, Pell equation III for Beyer *et al* 1986). Each zero belongs to a sequence of two different Pell equations.

Among the zeros with $i = j = 1$, there are pairs of zeros with the same value of N and t, two of them with $t = 2z + 1$ and six with $t = \frac{3}{2}z + 1$. More generally, with $r(t-1) = (r+1)z$, the condition reduces to

$$\left[x - 2r^2 + 1\right]\left[y - 2r^2 - 2r - 1\right] = r(r+1)(2r-1)(2r+1) \tag{29}$$

which defines for any value of r a finite family of pairs of zeros with the same value of t, z and a difference $2r+2$ for the other parameters. For $N \leqslant 2400$, there are five pairs out of the 12 pairs of zeros related to $r=3$. At least eight polynomial families of pairs can be extracted from these zeros. They are

$$\begin{aligned} x &= 2r^2 - 1 + p(r) \\ y &= 2r^2 + 2r + 1 + q(r) \\ p(r)q(r) &= r(r+1)(2r-1)(2r+1) \end{aligned} \tag{30}$$

where $p(r)$ and $q(r)$ are any polynomial of r including unity. There are only two such pairs with $i \neq j$ for $N \leqslant 2400$.

Case 4: $(x-1)y = (u+1)(v+1)$. There are 599 such zeros (105, 154, 340). The condition (11) can be written as

$$(u+1)(v+1)(x+y+u+v+1)[uv(x+y+u+v) - z(z+2y+2u+2v+1)] = 0 . \tag{31}$$

With $x = gik+1, y = gjl, u = gjk-1, v = gil-1$ there are two Pell equations which differ from the Pell equations (26) of case 3 by

$$i \Leftrightarrow j \qquad z \Rightarrow z + g(ik - jk + il + jl) - 2 . \tag{32}$$

Therefore, for any zero of case 3 with $x_3, \ldots$ there is a zero of case 4 with $x_4, \ldots$ such that

$$\begin{array}{lll} x_4 = u_3 + 1 & y_4 = v_3 + 1 & z_4 = z_3 - x_3 - y_3 + u_3 - v_3 + 1 \\ u_4 = x_3 - 1 & v_4 = y_3 - 1 & t_2 = t_3 - x_3 - y_3 + u_3 - v_3 + 1 . \end{array} \tag{33}$$

However, five pairs of zeros of case 3 give the same zero of case 4. The finite families of the two cases are equivalent only if $i = j = 1$. For 40 zeros, the finite families generate no others but one of these families involves 354 zeros.

There are 130 zeros (31, 29, 70) with $i = j = 1$. They have two linear conditions between quantum numbers: $x = u+2$ and $y = v+1$. In this case the 6-j coefficient can be written as

$$\begin{Bmatrix} a+b-2 & a & b \\ c & c & c+\frac{1}{2} \end{Bmatrix} \qquad \begin{array}{lll} x = a+\frac{1}{2} & y = b & z = -a-b+2c+\frac{5}{2} \\ u = a-\frac{3}{2} & v = b-1 & t = a+b+2c+\frac{3}{2} . \end{array} \tag{34}$$

This is case 1, Pell equation I and case 7, Pell equation III for Beyer *et al* 1986) with $[4c+2b+1]$ instead of $[4c-2a+3]$ and $[8c+4b+2]$ instead of $[8c-4a+6]$, respectively, in equations (27). There are pairs similar to those defined in (29) and (30). Here, finite families can be defined with

$$\left[x - 2r^2 - 2r - 1\right]\left[y - 2(r+1)^2\right] = r(r+1)(2r+1)(2r+3) \tag{35}$$

instead of (29), but there is no relation from pair to pair between cases 3 and 4. There are also only two such pairs with $i \neq j$ for $N \leqslant 2400$. The correspondence between cases 3 and 4 is

$$\begin{Bmatrix} a+b-2 & a & b \\ c & c & c-\frac{1}{2} \end{Bmatrix}_{(3)} = 0 \Rightarrow \begin{Bmatrix} a+b-2 & a & b \\ c-b+\frac{1}{2} & c-b+\frac{1}{2} & c-b+1 \end{Bmatrix}_{(4)} = 0 . \tag{36}$$

There are 1623 zeros belonging to one of these four cases for $N \leqslant 2400$ because some of them belong to more than one case. It is almost 10% of the total number. Among them, 370 with $i = j = 1$ belong to six of the cases studied by Beyer *et al* (1986). Almost as many (334, that is 54, 70, 116 and 130 for cases 1 to 4) with $i = 1, j = 2$ need only a minor modification of the Pell equations of these authors.

3.2. Two relations between quantum numbers

All the cases of the Pell equation found by Beyer *et al* (1986) request two relations between the quantum numbers. Three of them cannot be generalized to deal with only one relation. Searching for polynomial families, we found a fourth case with a quadratic condition.

Case a: $x = u + 1$, $y = v + 1$. There are 39 such zeros (15, 11, 13). The generalization to the unique condition $xy = (u+1)(v+1)$ gives only four other zeros for $N \leqslant 2400$. It is the most symmetric case because

$$\begin{Bmatrix} a+b-2 & a & b \\ c & c & c \end{Bmatrix} \qquad \begin{matrix} x = a & y = b & z = -a-b+2c+2 \\ u = a-1 & v = b-1 & t = a+b+2c+1\,. \end{matrix} \tag{37}$$

The condition for a zero is

$$2(u+1)(v+1)(u+v+1)[uv(2u+2v+3) - 2(u+v+1)z - z^2] = 0 \tag{38}$$

and can be written as the Pell equation

$$[4c+2]^2 - (2b-1)[2a+b-1]^2 = -(b-1)(2b^2+b+1) \tag{39}$$

(case 2 and Pell equation II from Beyer *et al* 1986). There is a second Pell equation obtained by the exchange of a and b. So, each zero belongs to two different sequences of solutions of a Pell equation. Each zero also belongs to six different finite families.

Case b: $x = u - 1$, $y = v$. There are 33 such zeros (8, 12, 13). The quantum numbers of the lower row are all different:

$$\begin{Bmatrix} a+b-2 & a & b \\ c+\frac{1}{2} & c & c-\frac{1}{2} \end{Bmatrix} \qquad \begin{matrix} x = a-1 & y = b-\frac{1}{2} & z = -a-b+2c+\frac{3}{2} \\ u = a & v = b-\frac{1}{2} & t = a+b+2c+\frac{1}{2}\,. \end{matrix} \tag{40}$$

The condition for a zero is

$$2v(u+v)\big[u(u+1)(v+1)(2u+2v-1) + (2uv+3u+2v)z - (u-2)z^2\big] = 0\,. \tag{41}$$

This condition can be written as the Pell equation

$$\begin{aligned}[]
[(a-2)(4c-2a-4b+1) - 3(2b+1)]^2 - (a^2-1)(2a-1)[a+2b-1]^2 \\
= -a(a-2)^2(2a^2-a-2)
\end{aligned} \tag{42}$$

(case 8 and Pell equation IV for Beyer *et al* 1986). Each zero only belongs to a sequence of Pell equations. No finite family can be defined.

Case c: $x = u + 3$, $y = v + 2$. There are 38 such zeros (15, 7, 16). This case is similar to case b:

$$\begin{Bmatrix} a+b-2 & a & b \\ c-\frac{1}{2} & c & c+\frac{1}{2} \end{Bmatrix} \qquad \begin{matrix} x = a+1 & y = b+\frac{1}{2} & z = -a-b+2c+\frac{5}{2} \\ u = a-2 & v = b-\frac{3}{2} & t = a+b+2c+\frac{3}{2}\,. \end{matrix} \tag{43}$$

The condition for a zero is

$$\begin{aligned}
2(v+1)(u+v+3)\big[u(u+1)v(2u+2v+5) - (4u^2+6uv+15u+6v+12)z - uz^2\big] \\
= 0\,.
\end{aligned} \tag{44}$$

This condition can be written as the Pell equation

$$[(a-2)(4c+2a+4b+3)+3(2b+1)]^2-(a^2-1)(2a-1)[a+2b-1]^2 = -a(a-2)^2(2a^2-a-2) \tag{45}$$

(case 9 and Pell equation IV for Beyer *et al* 1986). Each zero only belongs to a sequence of two different Pell equations. No finite family can be defined. Due to the use of Pell equation IV, there is a correspondence:

$$\begin{Bmatrix} a+b-2 & a & b \\ c+\frac{1}{2} & c & c-\frac{1}{2} \end{Bmatrix}_{(b)} = 0 \Rightarrow \begin{Bmatrix} a+b-2 & a & b \\ c'-\frac{1}{2} & c' & c'+\frac{1}{2} \end{Bmatrix}_{(c)} = 0 \tag{46}$$

provided that

$$c' = c-(a+2b-1)\frac{2a-1}{2a-4} \tag{47}$$

is integer or half-integer, that is for 20 zeros in the 33 found for $N \leqslant 2400$ in case b and for 25 zeros in the 38 found in the same range for case c.

Case d: $y-x=v-u$, $x=2(v-u)^2-v-1$. Searching for families with a polynomial behaviour of all the parameters and trying the relations (13) and (14) on them, one can find families with quadratic behaviour related by (13) using x and v and by (14) using z and t. These zeros have $y-x=v-u$ and $y+u+1=2(y-x)^2$, that is two relations as in cases a to c. There are 211 such zeros for $N \leqslant 2400$ (47, 57, 107). Expressing u and v with $w=y-x$, the condition (11) becomes

$$2(2w^2-1)(4w^2-1)\Big[(w^2-x)(z+4w^2-1)+x^2-w^2\Big] \times\Big[(w^2-y)(z+4w^2-1)+y^2-w^2\Big]=0\,. \tag{48}$$

This family of zeros can be separated into a sub-families fulfilling the first condition with 105 zeros and a sub-family fulfilling the second condition with 106 zeros. The relation (13) with exchange of x with y and u and v increases or decreases the four parameters x, y, u and v by the same amount w (it fails for one zero of the second set, due to a value out of range). The relation (14) shifts from one condition to the other but fails for 69 of the first set (five due to values out of range and 64 due to non-integer values) and 51 zeros of the second set (due to non integer values).

The first condition can give zeros only if $x > w^2$. With two positive integers i and j without a common divisor such that $j(x-w^2)=iw$, w must be divisible by j, that is $w=jr$. The solution of the first condition is

$$x=r(j^2r+i) \qquad y=r(j^2r+i+j) \qquad z=\frac{1}{i}(j^2r-i)(j^2r^2-ir-1)\,. \tag{49}$$

As $y=x+w$, the solution of the second condition is, with $k=i+j$,

$$x=r(j^2r+k-j) \qquad y=r(j^2r+k) \qquad z=\frac{1}{k}(j^2r-k)(j^2r^2-kr-1)\,. \tag{50}$$

All the parameters are expressed by a polynomial in r, which is at most cubic. Many families with a quadratic behaviour of the parameter can be extracted with a third condition. All the values of r are not allowed. Using l for i of (49) and for k of (50), the allowed values of r are n_l+nl for any value of n and $0 \leqslant n_l < l$ are such that $n_l(jn_l-1)(jn_l+1)$ is a multiple of l. The number of values of n_l is 3^m, where m is the number of prime factors of l larger than 2 multiplied by 5 if l is divisible by 8, 3 if l is divisible only by 4 and 2 if l is divisible only by 2.

These cases include 317 zeros because four of them belong to more than one case. Among them, 11 zeros also belong to the case for which only one condition was necessary.

3.3. Three relations between quantum numbers

Using a polynomial expansion of the parameters in the condition (11), we obtain that families of zeros for degree 2 are at least quadratic. Linear families will be found only for degree 3. The simplest case is when one of the parameters (u, v) or (x, y, z) is constant. Writing the other parameters as $x = x_0 + x_1 r + x_2 r^2$, the higher degrees of r in condition (11) imply that a parameter of the other set should be constant or that two parameters of the other set should be linear in r. No family with a constant parameter among (x, y, z) and a constant parameter among (u, v) have been found.

There are quadratic families with a constant value of u. With $u = 3$, $y = x + 2$, $v = z - x + 2$, the condition for a zero is

$$(z+3)(z+4)(4z - x^2 - 5x + 14)(12z - x^2 - 5x + 24) = 0\,. \tag{51}$$

Each term gives a family of 16 zeros for $N \leqslant 2400$ with z quadratic in x, but the first one verifies the condition (31) and belongs to case 4. The relations (13) and (14) used on the second family give two similar quadratic families having eight zeros for $N \leqslant 2400$, verifying, respectively, the conditions (20) and (25). Details are given in table 2 in which these families are described using numbers 1 and 2 and the families obtained with relations (13) or (14) are 2*a* and 2*b*. Other quadratic families can be found with $u = 3$ and with $u = 8$. In this table, the quadratic families are separated into five groups: 1 to 6 with fixed value of u, 7 to 11 with fixed value of x, 12 to 19 with $z - y = v - u$, 20 to 30 with $z - y = v - u \pm 1$ and 31 to 40 miscellaneous. A last group gives cubic families. Already, three families of pairs discussed in equations (29)–(30) and (35) are cubic: there are those for which $p(r)$ and $q(r)$ are quadratic in (30). A family with cubic behaviour of some parameters has also be found, using relation (14) for the quadratic family 10.

A Pell equation has been found for the 6-j coefficients with c and $c \pm \frac{1}{2}$ in the lower row given by (40) and (43). But for the third disposition of these quantum numbers in the lower row:

$$\begin{Bmatrix} a+b-2 & a & b \\ c & c+\frac{1}{2} & c-\frac{1}{2} \end{Bmatrix} \qquad \begin{matrix} x = a - \frac{1}{2} & y = b + \frac{1}{2} & z = -a - b + 2c + 2 \\ u = a - \frac{1}{2} & v = b - \frac{3}{2} & t = a + b + 2c + 1 \end{matrix} \tag{52}$$

the condition (11) is

$$\begin{aligned} 2x(y-1)\big[&(x+1)(y-2)(x+y-1)(2x+2y-1) \\ &-(4x^2+4xy+x+3y-3)z - (x+y+1)z^2\big] = 0 \end{aligned} \tag{53}$$

and there is no Pell equation. With $x = u$, $y = v + 2$ there are 15 zeros $(8, 4, 3)$ for $N \leqslant 2400$. Among them, there are five pairs with the same N, z and t and with $x' = x + 1$, $y' = y - 1$. The condition gives a family of pairs:

$$x = r^2 + 2r - 1 \qquad y = (r+2)^2 \qquad z = (2r+1)(r^2 + 3r + 1) \tag{54}$$

with r starting from 1. The other five zeros with $u = x$ and $v = y - 2$ are for $N \leqslant 2400$ at $N = 91, 271, 277, 304$ and 1561. For $N \leqslant 120\,000$ there are 21 other pairs and only three

Table 2. Polynomial families with three conditions between quantum numbers. An identification i is given to each family. The number of zeros for $N \leqslant 2400$ is ν and the number of the other zeros with the same conditions is μ. The last part of the table gives cubic families. The five first parts give quadratic families for fixed u, fixed x, $z-y=v-u$, $z-y=v-u\pm 1$ and miscellaneous. All the values of the variable r are not allowed. For some families r can be positive and negative. For fixed value of u or of x the families which are subsets of more general cases are also given.

i	ν / μ	Condition 1 / Variable	Condition 2 / Equation	Condition 3 / Remark
1	16	$u=3$	$v=z-x+2$	$y=x+2$
	16	$r=x$	$4z=x^2+5x-14$	Subset of case 4
2	16	$u=3$	$v=z-x+2$	$y=x+2$
	16	$r=x$	$12z=x^2+5x-24$	Factorized with 1
2a	8	$u=3$	$v=z-x+2$	$y=3x-1$
	1	$r=x$	$4z=3x^2-x-10$	(13) on 2
2b	8	$u=3$	$v=z-3x-5$	$y=3x+7$
	1	$r=x$	$4z=3x^2+19x+20$	(14) on 2
3	13	$u=3$	$v=2z-x+3$	$y=x+2$
	1	$r=x$	$12z=x^2+5x-24$	
4	13	$u=3$	$v=2z-2x+4$	$y=3x+3$
	0	$r=x$	$4z=x^2+3x-10$	
4a	5	$u=3$	$3v=2z-6x+7$	$y=3x+3$
	0	$r=x$	$20z=9x^2+27x-70$	(14) on 4
5	17	$u=8$	$v=z-x+4$	$y=x+3$
	1	$r=x$	$12z=x^2+11x-60$	
6	12	$u=8$	$2v=2z-3x+9$	$y=2x+4$
	0	$2r=x-1$	$6z=x^2+8x-33$	
7	13	$x=4$	$v=u+2$	$z=2y+u+3$
	0	$r=u$	$4y=u^2+3u$	Subset of case 3
8	8	$x=4$	$v=3u+9$	$z=2y-u-7$
	0	$r=u$	$4y=3u^2+13u+14$	Subset of case 4
9	8	$x=4$	$v=3u+5$	$z=2y+2u+5$
	0	$r=u$	$4y=3u^2+7u$	
10	7	$x=4$	$v=3u+1$	$z=2y+5u+4$
	0	$r=u$	$4y=3u^2+u$	Subset of case 1
11	27	$x=9$	$v=u+3$	$z=y+2$
	0	$r=u$	$6y=u^2+7u$	
12	16	$x=2u+v+2$	$y=v+1$	$z=y-u+v$
	0	$r=2u-v$	$(2u-v)^2=5v+4$	
12a	16	$2x=6u+v+3$	$y=u+v$	$z=y-u+v$
	0	$2r=2u-v+1$	$(2u-v)^2=16u+12v+5$	(13) on 12
13	16	$v=3u+3$	$y=v+1$	$z=y-u+v$
	0	$r=x-3u$	$(x-3u)^2=13x-9u$	
14	12	$x=u$	$y=2u+v-1$	$z=y-u+v$
	0	$r=v-u$	$(v-u)^2=9u+9v-2$	
15	12	$v=4u+4$	$y=u+v+2$	$z=y-u+v$
	0	$r=x-2u$	$(x-2u)^2=7x-4u$	

other single zeros at $N=46033$, 55969 and 57754. These families are in table 3 with numbers 29 and 30. Relations (13) and (14) generate new families from each of them.

There are 15 zeros for $N \leqslant 2400$ with $x=u$, $2y=3v$ and $t=2z+1$, but they are members of a finite family which also includes another zero for $N=4296$. Many such finite families can be found. Without the finite families, the use of three conditions allowed

Table 2. Continued.

16	10	$2x = 4u + v + 5$	$y = -8u + v - 7$	$z = y - u + v$
	0	$2r = v - 12u - 11$	$(12u - v)^2 = -264u + 24v - 119$	
16*a*	10	$5x = 28u + v + 28$	$5y = 32u - v + 27$	$z = y - u + v$
	0	$5r = v - 12u - 12$	$(12u - v)^2 = -198u + 29v - 54$	(13) on 16
17	8	$x = 3v + 3$	$4y = 5v + 4$	$z = y - u + v$
	0	$8r = 3v - 4u + 4$	$(4u - 3v)^2 = 16v + 16$	
18	8	$x = 3v + 4$	$4y = 5v + 10$	$z = y - u + v$
	0	$4r = 3v - 4u + 4$	$(4u - 3v)^2 = 32u - 8v + 20$	
19	6	$x = 8u + 8$	$2y = 8u - v + 6$	$z = y - u + v$
	0	$2r = v - 4u$	$(4u - v)^2 = -14u + 6v$	
19*a*	6	$2x = v + 3$	$2y = 30u - 5v + 15$	$z = 8u + 8$
	0	$2r = v - 4u + 1$	$(4u - v)^2 = 10u + 9$	(13) on 19
19*b*	3	$x = 8u + 8$	$2y = 32u - 7v + 21$	$z = y - u + v$
	0	$2r = v - 4u - 1$	$5(4u - v)^2 = -94u + 24v - 27$	(13) on 19
20	21	$x = v + 1$	$y = 3v + 3$	$z = y - u + v - 1$
	0	$r = v - u$	$(u - v)^2 = u + v + 2$	
21	20	$x = v + 3$	$y = 3v + 4$	$z = y - u + v + 1$
	0	$r = v - u$	$(u - v)^2 = 5u - 3v$	
22	13	$x = 6u - 4v + 8$	$y = u + v + 4$	$z = y - u + v - 1$
	2	$r = v - u$	$2(u - v)^2 = 2u + v + 6$	
22*a*	13	$x = 2u + 2$	$y = 3v - u + 6$	$z = y - u + v + 1$
	0	$r = v - u$	$2(u - v)^2 = 7u - 4v$	(13) on 22
23	13	$2x = v + 2$	$y = 5v + 5$	$z = y + v - u - 1$
	0	$r = v - u$	$(u - v)^2 = 3u + 4$	
24	12	$x = 2v + 3$	$y = 2v + 5$	$z = y - u + v - 1$
	0	$r = v - u$	$2(u - v)^2 = 8u - 5v$	
25	12	$x = 2v$	$3y = 4u + 2v + 1$	$z = y + v - u + 1$
	0	$3r = v - u - 1$	$2(u - v)^2 = 14u + 13v + 7$	(13) on subset of case d
26	12	$2x = v + 4$	$y = 5v + 6$	$z = y + v - u + 1$
	0	$r = v - u$	$(u - v)^2 = 7u - 4v$	
27	10	$x = 4u$	$5y = 14u + v + 7$	$z = y + v - u - 1$
	0	$5r = v - 6u + 2$	$(6u - v)^2 = 64u + 31v + 16$	
28	8	$x = v + 1$	$y = 3v$	$z = y + v - u + 1$
	0	$3r = v - u - 1$	$3(u - v)^2 = -u + 7v + 2$	
29	8	$x = 3v - 4u - 1$	$y = 4v - 5u - 2$	$z = y + v - u - 1$
	0	$r = v - 2u$	$5(2u - v)^2 = -24u + 13v - 6$	
30	6	$x = 2u + 2$	$y = 9u - v + 6$	$z = y + v - u + 1$
	0	$2r = v - 4u$	$(4u - v)^2 = -14u + 6v$	

us to find 593 zeros, of which only 499 are new. With one, two and three relations, we found 2427 zeros, that is less than 15% of the total number of zeros for $N \leqslant 2400$.

3.4. Remark on zeros of $\begin{Bmatrix} 2 & a & a \\ c & b & b \end{Bmatrix}$

There are 21 such zeros with $z = 2$, $v = 1$, for $N \leqslant 2400$. They have been studied by Brudno (1987). He gave the solutions for $c = b$, in which case there is a Pell equation

$$4(2b+1)^2 - 3(2a+1)^2 + 11 = 0 \tag{55}$$

already studied by Brudno and Louck (1987) and for $c = 3b+1$, in which case the condition is the product of two Pell equations

$$\left[(2a+1)^2 - 12(2b+1)^2 - 1\right]\left[16(2b+1)^2 - 3(2a+1)^2 + 11\right] = 0 \tag{56}$$

Table 2. Continued.

31	19	$3x = u + 2v$	$y = 3u$	$3z = 8u + v + 6$
	0	$3r = v - u - 3$	$(u - v)^2 = 9u + 9v$	(13) on (49)
32	15	$x = 3u$	$2y = 3u + v - 5$	$z = 2u + v - 1$
	0	$2r = v - 3u - 1$	$(3u - v)^2 = 48u + 24v - 23$	
32*a*	15	$x = u$	$2y = 3u + v - 5$	$z = 2u + v - 1$
	1	$2r = v - 3u - 1$	$(3u - v)^2 = 48u + 24v - 23$	(14) on 32
33	12	$2x = 6u - 3v + 6$	$y = v + 2$	$z = 11u - 2v + 12$
	0	$r = v - u$	$(u - v)^2 = 3u + 4$	
34	12	$6x = u + 2v - 1$	$y = 5u$	$3z = 14u + v + 10$
	0	$3r = v - u + 1$	$(u - v)^2 = 14u + 13v - 13$	(13) for (50)
35	12	$2x = 2u + v - 3$	$y = v$	$z = 3u + 2v - 1$
	0	$2r = v - 2u - 1$	$(2u - v)^2 = 24u + 16v - 7$	
35*a*	12	$2x = 2u + v - 3$	$y = 2u$	$z = 3u + 2v - 1$
	0	$r = v - 2u$	$(2u - v)^2 = 24u + 16v - 7$	(14) on 35
36	10	$x = u + 1$	$y = 6u - v + 2$	$z = 8u + 3$
	0	$r = v - 2u$	$(2u - v)^2 = 3u + v + 2$	
37	9	$5x = 8u - v + 9$	$y = 2u + v + 2$	$5z = 27u + 16v + 31$
	0	$5r = 3u - v - 1$	$(3u - v)^2 = u + 8v + 9$	
37*a*	8	$3x = v + 4$	$3y = 30u - 5v + 13$	$z = 39u - 8v + 16$
	0	$3r = v - 3u - 2$	$(3u - v)^2 = 15u - 2v + 8$	(13) on 37
38	8	$3x = 5u + 3$	$y = 2u + 2$	$3z = 16u - 3v + 6$
	0	$9r = 3v - 4u - 6$	$(4u - 3v)^2 = 15u + 9v + 18$	
39	7	$2x = 3u$	$6y = 7u + 2v + 6$	$3z = 4u + 8v + 3$
	0	$3r = v - u$	$4(u - v)^2 = 21u + 24v + 18$	
40	7	$3x = 3u + 2v - 3$	$3y = 3u + 2v + 3$	$3z = 5v$
	0	$3r = v - 3u$	$2(3u - v)^2 = 54u + 27v + 45$	
10*a*	4	$x = 4$	$v = 3u + 1$	$4z = (y + 4u + 5)(3u + 4)$
		$r = u$	$2y = 3u^2 + 11u + 8$	(14) on 10
41	5	$x = u$	$2z = (v + u + 1)(v - u - 2)$	$y = v + 2$
		$2r = v - u - 3$	$(u - v)^2 = 2u + 2v + 7$	
41*a*	4	$x = u + 2$	$8z = (y - u - 10)(y + 3u + 4)$	$v = u + 2$
		$4r = y - v - 2$	$(y - u)^2 = 16y$	(13) on 40
41*b*	7	$x = u + 2$	$24z = (y - u - 2)(7u - 3y + 48)$	$v = u + 2$
		$4r = y - v - 2$	$(y - u)^2 = 16y$	(14) on 40
42	5	$x = u$	$2z = (v + u + 1)(v - u)$	$y = v + 2$
		$2r = v - u - 1$	$(u - v)^2 = 6u - 2v + 3$	
42*a*	4	$x = v$	$8z = (v - u)(u + 3v + 2)$	$y = v + 2$
		$4r = v - u + 6$	$(u - v)^2 = 12u + 4v + 12$	(13) on 41
42*b*	6	$x = v$	$24z = (v - u + 8)(5u - v + 6)$	$y = v + 2$
		$4r = v - u + 6$	$(u - v)^2 = 12u + 4v + 12$	(14) on 41

quoting only the solution of the first part of the condition. In fact, b is half-integer in all the solutions with $c = b$ and

$$\begin{Bmatrix} a & 2 & a \\ x+\frac{1}{2} & x+\frac{1}{2} & x+\frac{1}{2} \end{Bmatrix} = 0 \Rightarrow \begin{Bmatrix} a & 2 & a \\ \frac{1}{2}x & \frac{3}{2}x+1 & \frac{1}{2}x \end{Bmatrix} = 0. \qquad (57)$$

So, there is a solution with $c = 3b + 1$ corresponding to any solution with $c = b$.

4. Zeros of degree 3

There are 1939 zeros (261, 557, 1121) of degree 3. The maximum number of zeros for a given value of N is six and this happens for $N = 1834$. Searching for zeros with

Table 3. Largest linear families of zeros of degree 3. The number of members of the family for $N \leqslant 2400$ is ν. The first column gives the equation describing the family.

	ν	x	y	z	t	u	v	N
64	47	$3r+4$	$3r+6$	$5r+7$	$15r+22$	r	$3r+3$	$49r+64$
	47	$3r+4$	$3r+6$	$5r+10$	$15r+26$	r	$3r+4$	$49r+77$
	35	$2r+4$	$5r+6$	$8r+10$	$20r+26$	r	$4r+4$	$65r+77$
	35	$2r+3$	$5r+10$	$8r+16$	$20r+37$	r	$4r+6$	$65r+112$
	26	$4r+1$	$7r+2$	$10r+5$	$28r+9$	$2r-1$	$5r$	$91r+21$
	25	$4r+4$	$7r+7$	$10r+6$	$28r+22$	$2r$	$5r+3$	$91r+64$
	17	$3(4r-1)$	$7r$	$12r$	$42r-5$	$3r-2$	$2(4r-1)$	$137r-24$
	17	$4(3r+1)$	$7r+1$	$2(6r+1)$	$42r+8$	$3r-1$	$8r$	$137r+18$
	16	$5r$	$9r$	$21r-6$	$45r-8$	$3r-2$	$7r-2$	$145r-33$
	16	$5r+1$	$9r+1$	$21r+7$	$45r+11$	$3r-1$	$7r$	$145r+27$
66	9	$2r+1$	$32r+8$	$35r+7$	$80r+17$	$4r$	$7r-1$	$251r+45$
	8	$2r+2$	$32r+26$	$35r+20$	$80r+66$	$4r+1$	$7r+5$	$251r+199$
	6	$2r+1$	$39r+20$	$48r+20$	$104r+41$	$3r-1$	$12r+4$	$327r+121$
	6	$2r+2$	$39r+26$	$48r+30$	$104r+66$	$3r+1$	$12r+5$	$327r+199$
	5	$4r$	$63r-9$	$66r-9$	$154r-20$	$9r-2$	$12r-4$	$483r-71$
	4	$4r+1$	$63r+11$	$66r+11$	$154r+23$	$9r-1$	$12r+1$	$483r+64$
68	23	$2r+4$	$6r+7$	$15r+20$	$30r+40$	r	$6r+7$	$97r+122$
	23	$2r+3$	$6r+12$	$15r+26$	$30r+52$	r	$6r+9$	$97r+160$
	9	$8r$	$12r-3$	$39r-7$	$78r-14$	$3r-2$	$16r-4$	$253r-53$
	9	$8r+4$	$12r+1$	$39r+8$	$78r+16$	$3r-1$	$12r+2$	$253r+44$
70	19	$r+2$	$12r+18$	$18r+26$	$36r+52$	$2r+2$	$3r+2$	$113r+155$
	19	$r+2$	$12r+29$	$18r+29$	$36r+58$	$2r+1$	$3r+4$	$113r+174$
	8	$2r+1$	$30r+12$	$45r+17$	$90r+34$	$3r-1$	$10r+3$	$283r+99$
	7	$2r+2$	$30r+20$	$45r+29$	$90r+58$	$3r+1$	$10r+4$	$283r+174$

two identical parameters among (x, y, z) and among (t, u, v) gives 18 triplets (the other parameters are the three integer solutions of an algebraic equation of degree 3), 66 pairs and 10 sets of three zeros coupled two by two. There are also 11 pairs of zeros with the same values of (x, y, z) and 14 pairs of zeros with the same values of (t, u, v). Five equations like (12)–(14) can be written; they are of second order. If they are used on the 1939 zeros found, 1693 of them generate no other zeros and the others generate 80 pairs, 24 triplets, five sets of four and two sets of five zeros, involving 16 new zeros with $N > 2400$.

4.1. Zeros with $x = u+2$ and $y = v+2$

Among the 1939 zeros of degree 3, only 49 (13, 17, 19) have $x = u + 2$ and $y = v + 2$. These 6-j coefficients can be written with identical quantum numbers on the lower row:

$$\begin{Bmatrix} a+b-3 & a & b \\ c & c & c \end{Bmatrix} \qquad \begin{matrix} x = a & y = b & z = -a-b+2c+3 \\ u = a-2 & v = b-2 & t = a+b+2c+1\,. \end{matrix} \tag{58}$$

They have been studied by Brudno (1987) who gave a Pell equation and two expressions of a finite family. The condition for a zero is

$$4ab(a-1)(b-1)(a+b-1)(a+b-2) \\ \times\left[(2a-1)(2b-1)(2a+2b-3) - 3(4c+1)(4c+3)\right] = 0 \tag{59}$$

which can be written as the Pell equation

$$[6(2c+1)]^2 - 3(2a-1)[a+2b-2]^2 = -3(a-2)(2a^2-a+2) \tag{60}$$

and another one obtained by the exchange of a and b.

Since condition (59) is of the kind $f_1 f_2 f_3 = g_1 g_2$, where the f_i and the g_j are linear in a, b and c, we can introduce two positive numbers k and l without a common divisor, such that $kf_i = lg_j$. Elimination of c gives a quadratic equation for a and b which defines a finite family. So, because there are six choices for f_i, g_j, each zero belongs to six different finite families defined by two integers p and q without a common divisor. They are

$$\begin{aligned} &l(4c+2\pm 1) = k(2a+2b-3) \\ &(2l^2a-3k^2-l^2)(2l^2b-3k^2-l^2) = 3k(l\pm k)(3k^2\mp 3kl+2l^2) \\ &l(4c+2\pm 1) = k(2a-1) \\ &(3k^2+l^2-2l^2b)(2l^2a+2l^2b+3k^2-3l^2) = 3k(k\pm l)(3k^2\pm 3kl+2l^2) \end{aligned} \tag{61}$$

and the two others are obtained by the exchange of a and b in the last ones. Among the 294 families defined for the 49 zeros (which are not all different), 146 reduce to only one element but one involves 192 elements. By these relations, the zeros are related from to between four and 207 other zeros; the mean number of related zeros is 43 for $N \leqslant 2400$. The first two of these relations were given by Brudno (1987) using s such that $k(2s+1) = l$ in

$$a = \frac{3k^2+l^2}{2l^2} + p \qquad b = \frac{3k^2+l^2}{2l^2} + q \qquad pq = 3k(l\pm k)(3k^2\mp 3kl+2l^2). \tag{62}$$

A polynomial family in the parameter k is obtained by replacing p and q by two polynomials $p(k,l)$ and $q(k,l)$ such that $p(k,l)q(k,l)$ is the right member of each second equation (61). Similar results also hold for the conditions (15), (20), (25), (31) and (38).

It is quite easy to obtain the zeros of degree 3 with $x = u+2$ and $y = v+2$. There are 119 of them for $2400 < N \leqslant 12000$ and 111 for $12000 < N \leqslant 24000$. Their density decreases when N increases.

4.2. Zeros with $z - y = v - u$

There are 898 zeros (141, 250, 507) with $z-y = v-u$ (or $z-x = v-u$ or $y-x = v-u$) for $N \leqslant 2400$, that is 46.31% of their total number. Among them, there are 613 zeros with also $z = 3v-4u-2$ or $y = 3v-4u-2$. For them, the condition is

$$(2u-2v+1)(3u-3v+1)(3u-3v+2)f(x,u,v)[x(2u-v+2)+(u+2)v] = 0 \tag{63}$$

where $f(x,u,v)$ has 59 solutions for $N \leqslant 2400$. The 554 other zeros (78, 140, 326) can be written as

$$x = r+p+2 \qquad u = r \qquad v = \frac{2}{p}(r+1)(r+2)+2r+2 \tag{64}$$

for any positive value of p. The values of r are such that v is an integer: $r = n_p + pq$ with n_p ($0 \leqslant n_p < p$) such that $2(n_p+1)(n_p+2)$ is a multiple of p. The number of values of n_p is 2^m, where m is the number of prime factors of p larger than 2 multiplied by 4 if p is divisible by 4, multiplied by 2 if p is divisible by 2.

The value of v increases quadratically with r. However, it is possible to extract families for which all the parameters increase linearly, using $p = 2r+4$ or any parametrization of p and r such that v becomes linear. The 10 largest linear families are given in table 3. There is no general expression for these families and a zero can belong to more than one of them. With the restrictions $y = 2v-3u-2$ and $z = 3v-4u-2$, the search for zeros has been done for $N \leqslant 12\,000$ and gave a total of 4318 zeros. Among them, 146 are solutions of $f(x,u,v) = 0$ and 4172 belong to the family (64). The figures for $2400 < N \leqslant 6000$

are, respectively, 40 and 1230; for $6000 < N \leqslant 12\,000$, they are 47 and 2388, showing respectively a decrease and an increase of the density when N increases.

There are also 61 zeros with $z - y = v - u$ and $z = 3x + 2u + 3v + 7$. For them, the condition is

$$(3x + 3u + 3v + 7)(3x + 3u + 3v + 8)g(x, u, v) \times [5x(x-1) - 2(u - x + 2)(v - x + 2)] = 0 \tag{65}$$

where $g(x, u, v)$ has only one solution for $N = 1848$. The 60 other zeros (6, 16, 38) can be written as

$$x = r \qquad u = r + p - 2 \qquad v = \frac{5}{2p}r(r-1) + r - 2 \tag{66}$$

for any positive value of p and r such that the parameters are integers and in the allowed range. The values of r are such that v is an integer: $r = n_p + pq$ with n_p $(0 \leqslant n_p < p)$ such that $5(n_p + 1)(n_p + 2)$ is a multiple of $2p$. The number of values of n_p is 2^m, where m is the number of prime factors of p larger than 2 and not 5 multiplied by 10 if p is divisible by 25, multiplied by 5 if p is divisible only by 5.

As in the preceding case, the value of v increases quadratically with r. However, it is possible to extract families for which all the parameters increase linearly. Six linear families are given in table 3. There is no general expression for these families and a zero can belong to more than one of them. Computing all the zeros with $y = 3x+3u+2v+7$ and $z = 3x + 2u + 3v + 7$ for $N \leqslant 24\,000$, there are only two other solutions of $g(x, u, v) = 0$ for $N = 4370$ and $N = 8712$ and 1148 other zeros for (66), 455 below $N = 12\,000$ and 693 above.

4.3. Zeros with t =2z

There are 314 zeros (35, 96, 183) with $t = 2z$ for which $z(z-1)$ factorizes in the condition. Among them, 118 zeros (14, 32, 71) also have $z = 3(v - u) - 1$. For them the condition is

$$z(z-1)g(x, u, v)[2v(x - u - 2) - x(x + 4u + 3)] = 0 \tag{67}$$

where the complicated expression $g(x, u, v)$ has only one solution for $N = 1300$. The 117 other zeros can be written as

$$x = r \qquad u = r - p - 2 \qquad v = \frac{5}{2p}r(r-1) - 2r \tag{68}$$

for any positive value of p and r such that the parameters are integers and in the allowed range. The values of r are the same as for (66). Four linear families are given in table 3. The zeros with $t = 2z$ and $z = 3(v - u) - 1$ are quite easy to obtain; 780 zeros are found for $2400 < N \leqslant 12\,000$ and 1184 for $12\,000 < N \leqslant 24\,000$; all of them are given by (68).

Among the zeros with $t = 2z$, 107 (8, 33, 66) also have $3y = 2z + 2$ and $(y - 1)$ also factorizes in the condition which is

$$(y-1)z(z-1)h(x, u, v)[x(3u - x + 4) - (2x + v + 2)(u - x + 2)] = 0\,. \tag{69}$$

There are 18 zero solutions of $h(x, u, v)$ for $N \leqslant 2400$ and 89 (8, 25, 71) for the second expression. These 89 zeros can be written as

$$x = r \qquad u = r + p - 2 \qquad v = \frac{2}{p}r(r-1) + r - 2 \tag{70}$$

for any positive value of p and r such that the parameters are integers and in the allowed range. The values of r are similar to those of (64). Four linear families are given in table 3.

As for (64), (66) and (68), there is no general expression of these families and a zero can belong to more than one of them. For $N \leqslant 24\,000$, there are 1760 zeros with $t = 2z$, $3y = 2z+2$, of which 1712 belong to (70) and 48 are solutions of $h(x, u, v)$ in (69). These figures are 643 and 16 for $2400 < N \leqslant 12\,000$, 981 and 13 beyond 12 000.

4.4. Zeros with three relations between quantum numbers

Among the 59 zeros of $f(x, u, v)$ in (63), 12 have also $x = v+2$ and 10 have $v = 4u+7$. They are members of two quadratic families with three relations between quantum numbers. For them, $f(x, u, v)$ in (63) can be written, respectively, as

$$\begin{aligned} f(x,u,v) &= (v+1)(v+2)\big[(3u-v+3)^2+5u-5v+1\big]=0 \\ f(x,u,4u+7) &= (u+2)(4u+9)\big[(x-2u)^2+4u-17x\big]=0 \end{aligned} \tag{71}$$

and the solutions are

$$\begin{aligned} &x = v+2 \qquad y = 2v-3u-2 \qquad z = 3v-4u-2 \\ &u = \tfrac{1}{10}(r-2)(r+7) \qquad v = \tfrac{1}{10}(r+3)(3r-4) \end{aligned} \tag{72}$$

with $r = \pm 5q \pm 2$ and

$$\begin{aligned} &y = 5u+12 \qquad z = 8u+19 \qquad v = 4u+7 \\ &x = \tfrac{1}{60}(r+13)(r+17) \qquad u = \tfrac{1}{120}(r-17)(r+17) \end{aligned} \tag{73}$$

with $r = \pm 20q \pm 3$. If the relations $x = v+2$ and $v = 4u+7$ are replaced by the conditions (71), there are seven and one more zeros.

There are 19 zeros with $z-y = y-x = v-u$ for $N \leqslant 2400$. With $x = u+2$, six of them are members of a finite family relevant to the Pell equation. Among the others, six with $t = 4z-2$ and five with $t = 4x-2$ are members of a family for positive and negative values of the same parameter for which the condition is

$$\begin{aligned} &(z-1)(y+v)(y+v+1)(y+u)(y+u+1) \\ &\qquad \times \big[2(y+v)(u-v)+z-v-2\big]\big[2(v-u)^2-u-2\big]=0 \end{aligned} \tag{74}$$

and the solutions are

$$\begin{aligned} &x = 6u-4v+4t \qquad y = 5u-3v+4 \qquad z = 4u-2v+4 \\ &u = 2(r^2-1) \qquad v = 2r^2+r-2 \end{aligned} \tag{75}$$

with $r = 1$ to $r = 6$ and $r = -2$ to $r = -6$ for $N \leqslant 2400$.

A fourth infinite family can be found with $x = 4u+5$, $16y = 10u+9v+25$ and $4z = 14u+5v+21$ with the condition $3v = (2u+3)(4u-1)$; it involves only five zeros for $N \leqslant 2400$.

There are eight zeros with $x = 5$ among the 18 solutions of $h(x, u, v)$ in (69) and five zeros with $x = 5$ among the 59 solutions of $f(x, u, v)$ in (63) have $x = 5$. In these cases

$$\begin{aligned} h(5,u,v) &= [(u+1)(v+1)-5][(u-10)(v-10)-180]=0 \\ f(5,u,v) &= [(u+1)(v+6)+5][(u-10)(v+17)+180]=0\,. \end{aligned} \tag{76}$$

There are nine and six solutions, respectively, for these conditions and the zeros above $N = 2400$ occur for $N = 3952$ and $N = 2948$. Finally, for $t = 2z+2$, there are four zeros with $x = 5$, $y = 2u+2v+12$ and $z = 3u+3v+17$ with the condition $(u-3)(v-3) = 30$ and four zeros with $3x = 4u+12$, $z = 3v+7$ for which the condition $(u+6)(3v+48-14u) = 270$ gives two other solutions for $N = 3274$ and $N = 7290$.

Taking into account the Pell equation, the four families with two relations between quantum numbers and the seven families with three relations, we obtain 917 zeros, 47.29% for $N \leqslant 2400$; 130, 49.81% for $N \leqslant 600$; 248, 44.52% for $600 < N \leqslant 1200$; and 539, 48.08% for $1200 < N \leqslant 2400$.

5. Zeros of degree larger than 3

There are 217 zeros of degree 4 (53, 78, 86) for $N \leqslant 2400$. The maximum number of zeros for a given value of N is only two and happens 10 times. Of course, the first nine of the 10 zeros given by Brudno (1987) are in the list, but not the last one, which is of degree 3. Among the 18 zeros with $x = 7$ are three families of respectively five, three and four zeros with two other relations between the coefficients. These relations and the conditions for a zero are

$$\begin{array}{lll} y = 2u + 2v + 23 & z = 3u + 3v + 33 & (u-7)(v-7) = 140 \\ 2y = 3(u + v + 15) & z = 5(u + v + 13) & (u-7)(v-7) = 168 \\ y = -3u + 2v - 3 & z = -4u + 3v - 3 & (u-7)(v+17) = -140 . \end{array} \quad (77)$$

The two first families involve another zero for $N = 3138$ and $N = 4588$, respectively. These families account for 5.53% of the zeros found for $N \leqslant 2400$. We cannot say if the number of zeros is finite or not.

There are 58 zeros of degree 5 (20, 29, 9) for $N \leqslant 2400$. The condition for a zero is of degree 15. If $x - u$ and $y - v$ are from 2 to 6, this degree goes down: to degree 8 if these two differences are the same or one of them is 4; to degree 6 if both of them are 4. There are 12 zeros with $u = x - 6$ and $v = y - 6$, for which the condition factorizes into

$$4(x-2)(x-3)(y-2)(y-3)(x+y-4)(x+y-5)(x+y-6)f_1(x, y, z) = 0 . \quad (78)$$

Among them are eight zeros with $t = 2z$ ($z = 2x + 2y - 8$) and three zeros with $5z = 2x + 2y + 6$ for which the condition becomes, respectively,

$$f_1(x, y, 2x + 2y - 8) = (2x + 2y - 9)g_1(x, y)[(x-16)(y-16) - 180] = 0 \quad (79)$$

$$f_1(x, y, \tfrac{2}{5}[x + y + 3]) = (2x + 2y - 9)g_2(x, y)[(5x-32)(5y-32) - 504] = 0 . \quad (80)$$

The last factor of the first equation has a ninth solution for $N = 2704$. No solutions of $g_1(x, y)$ or $g_2(x, y)$ have be found for $N \leqslant 10^6$. There are only two other zeros with $u = x - 6$, $v = y - 6$ at $N = 631$ and $N = 5898$ for $N \leqslant 9000$. There are also 12 zeros with $u = x - 6$ and $v = y - 4$, for which the condition factorizes into

$$8(x-2)(x-3)y(y-1)(y-2)(x+y-3)(x+y-4)f_2(x, y, z) = 0 . \quad (81)$$

For six of them (including the first zero of the previous family), $z = 2y - x + 2$ and for five of them $z = 3x + 2y - 14$. For these

$$f_2(x, y, 2y - x + 2) = (2y-3)g_3(x, y)[(x + y + 10)(16 - x) - 180] = 0 \quad (82)$$

$$f_2(x, y, z) = (x + y - 5)(x + y - 6)g_4(x, y)[(x-16)(2y-33) - 360] = 0 . \quad (83)$$

The last factor of the second equation also has a sixth solution for $N = 6162$. No solution of $g_3(x, y) = 0$ has been found for $y \leqslant 10^6$, and $g_4(x, y) = 0$ has only one solution for $x = 75$ and $N = 2984$ for $x \leqslant 10^6$. There is only one other zero with $u = x - 6$, $v = y - 4$ at $N = 334$ for $N \leqslant 7500$. The condition for a zero is quite simple with $u = x - 4$ and $v = y - 4$, but gives only one zero at $N = 3054$ for $N \leqslant 24\,000$. All the zeros obtained for degree 5 are given in table 4. With 21 zeros (36.21% of the total number) given by four finite families, it seems that the total number of zeros is finite for this degree.

Table 4. Zeros of 6-j coefficients of degree $n = 5$. The values of a, b, c, d, e and f are chosen to exhibit the maximum number of almost identical values in the lower row and correspond to some permutation of the Y. Five zeros found beyond $N = 2400$ are added to the table. In the first column, 1, 2, 3 and 4 indicate zeros given by the relations (79), (80), (82) and (83), respectively; 5 stands for 2 and 3. This table is complete for $N = 2(a+b+c+d+e+f) \leqslant 2400$.

	N	X_a	X_b	X_c	Y_a	Y_b	Y_c	a	b	c	d	e	f
5	143	$\frac{53}{2}$	$\frac{47}{2}$	$\frac{43}{2}$	$\frac{7}{2}$	$\frac{7}{2}$	$\frac{3}{2}$	15	11	9	$\frac{23}{2}$	$\frac{25}{2}$	$\frac{25}{2}$
2	184	35	34	23	9	5	−2	20	16	18	7	16	16
3	212	40	36	30	10	6	0	25	21	15	15	15	15
	215	$\frac{81}{2}$	$\frac{73}{2}$	$\frac{61}{2}$	$\frac{23}{2}$	$\frac{11}{2}$	$-\frac{1}{2}$	26	21	15	$\frac{29}{2}$	$\frac{31}{2}$	$\frac{31}{2}$
	244	45	41	36	19	15	−12	$\frac{33}{2}$	13	$\frac{17}{2}$	$\frac{57}{2}$	28	$\frac{55}{2}$
	249	$\frac{91}{2}$	$\frac{83}{2}$	$\frac{75}{2}$	$\frac{25}{2}$	$\frac{13}{2}$	$\frac{9}{2}$	29	24	21	$\frac{33}{2}$	$\frac{35}{2}$	$\frac{33}{2}$
3	320	61	56	43	21	10	−3	29	23	12	32	33	32
	334	66	53	48	12	8	5	$\frac{71}{2}$	$\frac{45}{2}$	18	$\frac{61}{2}$	$\frac{61}{2}$	30
	341	$\frac{127}{2}$	$\frac{113}{2}$	$\frac{101}{2}$	$\frac{53}{2}$	$\frac{45}{2}$	$-\frac{31}{2}$	$\frac{53}{2}$	24	17	$\frac{79}{2}$	$\frac{79}{2}$	$\frac{77}{2}$
	348	67	60	47	26	23	−19	24	17	12	43	43	35
2	348	69	69	36	24	11	−9	40	39	6	29	30	30
	461	$\frac{163}{2}$	$\frac{159}{2}$	$\frac{139}{2}$	$\frac{115}{2}$	$\frac{23}{2}$	$-\frac{23}{2}$	$\frac{93}{2}$	$\frac{91}{2}$	6	35	34	$\frac{127}{2}$
	480	93	85	62	25	20	−1	59	42	41	34	42	21
	502	96	85	70	34	19	−4	46	33	18	50	52	52
3	506	97	91	65	41	17	−9	44	37	12	53	54	54
	509	$\frac{195}{2}$	$\frac{185}{2}$	$\frac{129}{2}$	$\frac{87}{2}$	$\frac{31}{2}$	$-\frac{19}{2}$	44	$\frac{77}{2}$	$\frac{21}{2}$	$\frac{107}{2}$	54	54
	512	100	91	65	42	15	−11	71	53	27	29	38	38
	566	106	92	85	54	7	0	80	46	46	26	46	39
	577	$\frac{223}{2}$	$\frac{197}{2}$	$\frac{157}{2}$	$\frac{57}{2}$	$\frac{47}{2}$	$\frac{7}{2}$	70	61	41	$\frac{83}{2}$	$\frac{75}{2}$	$\frac{75}{2}$
	592	107	98	91	49	41	−18	74	40	70	33	58	21
	605	$\frac{223}{2}$	$\frac{197}{2}$	$\frac{185}{2}$	$\frac{57}{2}$	$\frac{47}{2}$	$\frac{35}{2}$	70	61	55	$\frac{83}{2}$	$\frac{75}{2}$	$\frac{75}{2}$
	621	$\frac{239}{2}$	$\frac{201}{2}$	$\frac{181}{2}$	$\frac{65}{2}$	$\frac{39}{2}$	$\frac{19}{2}$	76	60	50	$\frac{87}{2}$	$\frac{81}{2}$	$\frac{81}{2}$
	630	111	107	97	71	59	−47	32	89	78	79	18	19
	630	111	108	96	72	59	−48	85	30	84	26	78	12
	631	$\frac{251}{2}$	$\frac{225}{2}$	$\frac{155}{2}$	$\frac{79}{2}$	$\frac{21}{2}$	$\frac{9}{2}$	68	54	19	$\frac{115}{2}$	$\frac{117}{2}$	$\frac{117}{2}$
	645	$\frac{251}{2}$	$\frac{247}{2}$	$\frac{147}{2}$	$\frac{111}{2}$	$\frac{77}{2}$	$-\frac{65}{2}$	$\frac{181}{2}$	81	$\frac{41}{2}$	35	$\frac{85}{2}$	53
	663	$\frac{257}{2}$	$\frac{253}{2}$	$\frac{153}{2}$	$\frac{103}{2}$	$\frac{23}{2}$	$\frac{3}{2}$	70	89	39	$\frac{117}{2}$	$\frac{75}{2}$	$\frac{75}{2}$
	669	$\frac{239}{2}$	$\frac{229}{2}$	$\frac{201}{2}$	$\frac{167}{2}$	$\frac{39}{2}$	$-\frac{35}{2}$	51	67	92	$\frac{137}{2}$	$\frac{95}{2}$	$\frac{17}{2}$
	698	132	128	89	72	32	−29	$\frac{103}{2}$	48	$\frac{17}{2}$	$\frac{161}{2}$	80	$\frac{161}{2}$
1	702	132	111	108	54	51	−28	52	30	27	80	81	81
1	715	$\frac{269}{2}$	$\frac{231}{2}$	$\frac{215}{2}$	$\frac{115}{2}$	$\frac{99}{2}$	$-\frac{57}{2}$	53	33	25	$\frac{163}{2}$	$\frac{165}{2}$	$\frac{165}{2}$
1	728	137	119	108	60	49	−29	54	35	24	83	84	84
4	816	153	136	119	73	58	−39	112	39	23	96	97	96
1	819	$\frac{309}{2}$	$\frac{279}{2}$	$\frac{231}{2}$	$\frac{147}{2}$	$\frac{99}{2}$	$-\frac{65}{2}$	61	45	21	$\frac{187}{2}$	$\frac{189}{2}$	$\frac{189}{2}$
	835	$\frac{313}{2}$	$\frac{295}{2}$	$\frac{227}{2}$	$\frac{171}{2}$	$\frac{107}{2}$	$-\frac{89}{2}$	56	47	14	$\frac{201}{2}$	$\frac{201}{2}$	$\frac{199}{2}$
4	877	$\frac{327}{2}$	$\frac{289}{2}$	$\frac{261}{2}$	$\frac{165}{2}$	$\frac{133}{2}$	$-\frac{95}{2}$	58	39	24	$\frac{211}{2}$	$\frac{211}{2}$	$\frac{213}{2}$
1	884	167	153	122	82	51	−35	66	51	20	101	102	102
3	887	$\frac{341}{2}$	$\frac{327}{2}$	$\frac{219}{2}$	$\frac{167}{2}$	$\frac{63}{2}$	$-\frac{45}{2}$	74	66	13	$\frac{195}{2}$	$\frac{195}{2}$	$\frac{193}{2}$
	901	$\frac{327}{2}$	$\frac{311}{2}$	$\frac{263}{2}$	$\frac{195}{2}$	$\frac{81}{2}$	$-\frac{49}{2}$	$\frac{261}{2}$	98	$\frac{107}{2}$	33	$\frac{115}{2}$	78
1	988	187	174	133	95	54	−39	74	60	19	113	114	114
	1010	182	169	154	112	100	−81	$\frac{101}{2}$	$\frac{69}{2}$	21	$\frac{263}{2}$	$\frac{269}{2}$	133
	1025	$\frac{389}{2}$	$\frac{351}{2}$	$\frac{285}{2}$	$\frac{147}{2}$	$\frac{75}{2}$	$\frac{5}{2}$	134	$\frac{213}{2}$	$\frac{145}{2}$	$\frac{121}{2}$	69	70
	1030	189	171	155	111	93	−77	56	39	22	133	132	133
	1083	$\frac{395}{2}$	$\frac{357}{2}$	$\frac{331}{2}$	$\frac{195}{2}$	$\frac{169}{2}$	$-\frac{91}{2}$	76	138	125	$\frac{243}{2}$	$\frac{81}{2}$	$\frac{81}{2}$

Table 4. Continued.

	N	X_a	X_b	X_c	Y_a	Y_b	Y_c	a	b	c	d	e	f
4	1094	204	188	155	113	78	−62	71	55	21	133	133	138
	1113	$\frac{427}{2}$	$\frac{405}{2}$	$\frac{281}{2}$	$\frac{209}{2}$	$\frac{93}{2}$	$-\frac{63}{2}$	91	78	18	$\frac{245}{2}$	$\frac{249}{2}$	$\frac{245}{2}$
	1122	220	215	126	100	50	−39	160	88	88	60	127	38
1	1170	222	210	153	117	60	−46	88	135	135	134	75	18
	1199	$\frac{445}{2}$	$\frac{385}{2}$	$\frac{369}{2}$	$\frac{195}{2}$	$\frac{53}{2}$	$\frac{41}{2}$	160	$\frac{219}{2}$	$\frac{205}{2}$	$\frac{125}{2}$	83	82
	1260	237	210	183	59	55	32	146	89	62	91	121	121
	1424	286	274	152	84	50	−4	185	139	51	101	135	101
1	1547	$\frac{589}{2}$	$\frac{567}{2}$	$\frac{391}{2}$	$\frac{323}{2}$	$\frac{147}{2}$	$-\frac{121}{2}$	117	105	17	$\frac{355}{2}$	$\frac{357}{2}$	$\frac{357}{2}$
4	1580	295	281	214	176	107	−93	101	87	19	194	194	195
	1594	299	289	209	179	101	−91	104	94	15	195	195	194
	1821	$\frac{717}{2}$	$\frac{703}{2}$	$\frac{401}{2}$	$\frac{311}{2}$	$\frac{47}{2}$	$\frac{9}{2}$	191	$\frac{347}{2}$	$\frac{45}{2}$	$\frac{335}{2}$	178	178
	1905	$\frac{781}{2}$	$\frac{771}{2}$	$\frac{353}{2}$	$\frac{311}{2}$	$\frac{117}{2}$	$-\frac{105}{2}$	$\frac{449}{2}$	219	$\frac{21}{2}$	116	$\frac{333}{2}$	116
3	2048	394	386	244	216	76	−66	164	155	14	230	231	230
4	2335	$\frac{873}{2}$	$\frac{847}{2}$	$\frac{615}{2}$	$\frac{543}{2}$	$\frac{307}{2}$	$-\frac{281}{2}$	148	135	18	$\frac{577}{2}$	$\frac{577}{2}$	$\frac{579}{2}$
1	2704	517	507	328	296	117	−105	206	195	16	311	312	312
	2984	566	498	428	280	212	−142	212	143	74	354	355	354
	3054	636	634	257	243	136	−134	386	384	7	250	250	250
	5898	1142	1110	697	621	208	−174	484	451	38	658	659	659
	6162	1154	1142	785	751	392	−380	387	375	17	767	767	768

Only 14 zeros of degree larger than 5 were found in the search for $N \leqslant 1200$. After continuation of the search for the degrees $n \leqslant 8$ and $N \leqslant 2400$:

- for $n = 6$, there are six zeros for $N \leqslant 1200$ and only one more at $N = 1581$;
- for $n = 7$, there are four zeros for $N \leqslant 1200$ but two more at $N = 1364$ and $N = 1701$;
- for $n = 8$, there are three zeros for $N \leqslant 1200$ and no more up to $N = 2400$;
- for $n = 9$, there is only one zero for $N = 411$ (we did not search beyond $N = 1200$);
- for $n > 9$, no zeros have been found for $N \leqslant 1200$.

All the zeros of degree larger than 5 are given in table 5.

6. Conclusions

The behaviour of the zeros of 6-j coefficients is similar to that found by Raynal *et al* (1992) for the 3-j coefficients of which the order is larger than the degree.

Here, there is nothing like the order introduced for the 3-j coefficients. The zeros of 6-j coefficients appear as a single population, whereas the zeros of 3-j coefficients were the mixture of two populations, these three populations showing similar properties of fast decrease of their number with increasing degree or order.

The number of zeros of degrees 1, 2 and 3 is infinite: this is proven via the use of the Pell equation in the last two cases. A Pell equation for the 6-j coefficient of degree 3 can only be used for 6-j which can be written with identical quantum numbers in the lower row; for degree 2, Pell equations deal primarily with similar coefficients. Four cases obtained by Beyer *at al* (1986) with two linear relations between quantum numbers can be generalized to deal with a single relation and we obtained another case of factorization, with two quadratic relations between quantum numbers, which do not give a Pell equation but give very simple polynomial solutions. Quite surprisingly, the 6-j coefficients of degree 3

Table 5. Zeros of 6-j coefficients of degree $6 \leqslant n \leqslant 8$ for $N = 2(a+b+c+d+e+f) \leqslant 2400$ and of degree $n > 8$ for $N \leqslant 1200$. The values of a, b, c, d, e and f are chosen to exhibit the maximum number of almost identical values in the lower row and correspond to some permutation of the Y.

N	n	X_a	X_b	X_c	Y_a	Y_b	Y_c	a	b	c	d	e	f
338	6	67	53	49	12	7	4	$\frac{79}{2}$	$\frac{49}{2}$	21	$\frac{55}{2}$	$\frac{57}{2}$	28
378	6	69	63	57	18	12	9	$\frac{87}{2}$	$\frac{75}{2}$	33	$\frac{51}{2}$	$\frac{51}{2}$	24
636	6	125	123	70	55	3	2	$\frac{123}{2}$	60	$\frac{15}{2}$	$\frac{127}{2}$	63	$\frac{125}{2}$
688	6	124	119	101	79	59	54	35	30	11	89	89	90
801	6	$\frac{305}{2}$	$\frac{267}{2}$	$\frac{229}{2}$	$\frac{93}{2}$	$\frac{55}{2}$	$\frac{19}{2}$	$\frac{199}{2}$	$\frac{161}{2}$	62	53	53	$\frac{105}{2}$
1033	6	$\frac{383}{2}$	$\frac{335}{2}$	$\frac{315}{2}$	$\frac{181}{2}$	$\frac{41}{2}$	$\frac{21}{2}$	141	94	84	$\frac{101}{2}$	$\frac{147}{2}$	$\frac{147}{2}$
1581	6	$\frac{587}{2}$	$\frac{565}{2}$	$\frac{429}{2}$	$\frac{353}{2}$	$\frac{213}{2}$	$\frac{183}{2}$	101	88	19	$\frac{385}{2}$	$\frac{389}{2}$	$\frac{391}{2}$
382	7	72	63	56	16	11	6	44	37	31	28	26	25
590	7	106	98	91	48	40	19	77	69	36	29	29	55
617	7	$\frac{235}{2}$	$\frac{201}{2}$	$\frac{181}{2}$	$\frac{61}{2}$	$\frac{39}{2}$	$\frac{19}{2}$	74	55	50	$\frac{87}{2}$	$\frac{81}{2}$	$\frac{81}{2}$
701	7	$\frac{263}{2}$	$\frac{257}{2}$	$\frac{181}{2}$	$\frac{141}{2}$	$\frac{65}{2}$	$\frac{59}{2}$	51	48	10	$\frac{161}{2}$	$\frac{161}{2}$	$\frac{161}{2}$
1364	7	265	215	202	75	37	26	151	70	88	114	145	114
1701	7	$\frac{659}{2}$	$\frac{619}{2}$	$\frac{423}{2}$	$\frac{279}{2}$	$\frac{79}{2}$	$\frac{3}{2}$	164	135	36	$\frac{331}{2}$	$\frac{349}{2}$	$\frac{351}{2}$
384	8	71	64	57	16	12	6	$\frac{87}{2}$	38	$\frac{63}{2}$	$\frac{55}{2}$	26	$\frac{51}{2}$
512	8	97	92	67	41	15	10	$\frac{87}{2}$	$\frac{77}{2}$	13	$\frac{107}{2}$	$\frac{107}{2}$	54
1116	8	216	214	128	87	22	1	119	$\frac{213}{2}$	$\frac{41}{2}$	97	$\frac{215}{2}$	$\frac{215}{2}$
411	9	$\frac{151}{2}$	$\frac{147}{2}$	$\frac{113}{2}$	$\frac{41}{2}$	$\frac{29}{2}$	$\frac{3}{2}$	48	44	29	$\frac{55}{2}$	$\frac{59}{2}$	$\frac{55}{2}$

are easier to handle: there are four other cases in which two simple linear relations between quantum numbers give a factorization of the condition. They do not give Pell equations but give results similar to those obtained for degree 2 with simpler expressions: linear instead of quadratic, quadratic instead of cubic. These cases of factorization account for almost half of the zeros. This situation can be compared to the one of the zeros of 3-j coefficients in which four fifths of the zeros are accounted for by degree 3.

For the zeros of degrees 4 and 5, we obtained cases of factorization with three simple relations between quantum numbers giving finite families. They are special cases of Pell equations $x^2 - Ny^2 = D$ in which N is a square. It is an indication, but not a proof, that the total number of zeros is finite. The scarcity of the zeros found for higher degrees forbids any further study.

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